



The impenetrable proof

Shinichi Mochizuki claims to have solved one of the most important problems in mathematics. The trouble is, hardly anyone can work out whether he's right.

By Davide Castelvecchi

Sometime on the morning of 30 August 2012, Shinichi Mochizuki quietly posted four papers on his website.

The papers were huge — more than 500 pages in all — packed densely with symbols, and the culmination of more than a decade of solitary work. They also had the potential to be an academic bombshell. In them, Mochizuki claimed to have solved the *abc* conjecture, a 27-year-old problem in number theory that no other mathematician had even come close to solving. If his proof was correct, it would be one of the most astounding achievements of mathematics this century and would completely revolutionize the study of equations with whole numbers.

Mochizuki, however, did not make a fuss about his proof. The respected mathematician, who works at Kyoto University's Research Institute for Mathematical Sciences (RIMS) in Japan, did not even announce his work to peers around the world. He simply posted the papers, and waited for the world to find out.

Probably the first person to notice the papers was Akio Tamagawa, a colleague of Mochizuki's at RIMS. He, like other researchers, knew

that Mochizuki had been working on the conjecture for years and had been finalizing his work. That same day, Tamagawa e-mailed the news to one of his collaborators, number theorist Ivan Fesenko of the University of Nottingham, UK. Fesenko immediately downloaded the papers and started to read. But he soon became “bewildered”, he says. “It was impossible to understand them.”

Fesenko e-mailed some top experts in Mochizuki's field of arithmetic geometry, and word of the proof quickly spread. Within days, intense chatter began on mathematical blogs and online forums (see *Nature* <http://doi.org/725>; 2012). But for many researchers, early elation about the proof quickly turned to scepticism. Everyone — even those whose area of expertise was closest to Mochizuki's — was just as flummoxed by the papers as Fesenko had been. To complete the proof, Mochizuki had invented a new branch of his discipline, one that is astonishingly abstract even by the standards of pure maths. “Looking at it, you feel a bit like you might be reading a paper from the future, or from outer space,” number theorist Jordan Ellenberg, of the University of Wisconsin–Madison, wrote on his blog a few days after the paper appeared.

Three years on, Mochizuki's proof remains in mathematical limbo — neither debunked nor accepted by the wider community. Mochizuki has estimated that it would take a maths graduate student about 10 years to be able to understand his work, and Fesenko believes that it would take even an expert in arithmetic geometry some 500 hours. So far, only four mathematicians say that they have been able to read the entire proof.

Adding to the enigma is Mochizuki himself. He has so far lectured about his work only in Japan, in Japanese, and despite being fluent in English, he has declined invitations to talk about it elsewhere. He does not speak to journalists; several requests for an interview for this story went unanswered. Mochizuki has replied to e-mails from other mathematicians and been forthcoming to colleagues who have visited him, but his only public input has been sporadic posts on his website. In December 2014, he wrote that to understand his work, there was a “need for researchers to deactivate the thought patterns that they have installed in their brains and taken for granted for so many years”. To mathematician Lieven Le Bruyn of the University of Antwerp in Belgium, Mochizuki's attitude sounds defiant. “Is it just me,” he wrote on his blog earlier this year, “or is Mochizuki really sticking up his middle finger to the mathematical community?”

Now, that community is attempting to sort the situation out. In

December, the first workshop on the proof outside of Asia will take place in Oxford, UK. Mochizuki will not be there in person, but he is said to be willing to answer questions from the workshop through Skype. The organizers hope that the discussion will motivate more mathematicians to invest the time to familiarize themselves with his ideas — and potentially move the needle in Mochizuki's favour.

In his latest verification report, Mochizuki wrote that the status of his theory with respect to arithmetic geometry “constitutes a sort of faithful miniature model of the status of pure mathematics in human society”. The trouble that he faces in communicating his abstract work to his own discipline mirrors the challenge that mathematicians as a whole often face in communicating their craft to the wider world.

Primal importance

The *abc* conjecture refers to numerical expressions of the type $a + b = c$. The statement, which comes in several slightly different versions, concerns the prime numbers that divide each of the quantities a , b and c . Every whole number, or integer, can be expressed in an essentially unique way as a product of prime numbers — those that cannot be further factored out into smaller whole numbers: for example, $15 = 3 \times 5$ or $84 = 2 \times 2 \times 3 \times 7$. In principle, the prime factors of a and b have no connection to those of their sum, c . But the *abc* conjecture links them together. It presumes, roughly, that if a lot of small primes divide a and b then only a few, large ones divide c .

This possibility was first mentioned in 1985, in a rather off-hand remark about a particular class of equations by French mathematician Joseph Oesterlé during a talk in Germany. Sitting in the audience was David Masser, a fellow number theorist now at the University of Basel in Switzerland, who recognized the potential importance of the conjecture, and later publicized it in a more general form. It is now credited to both, and is often known as the Oesterlé–Masser conjecture.

A few years later, Noam Elkies, a mathematician at Harvard University in Cambridge, Massachusetts, realized that the *abc* conjecture, if true, would have profound implications for the study of equations concerning whole numbers — also known as Diophantine equations after Diophantus, the ancient-Greek mathematician who first studied them.

Elkies found that a proof of the *abc* conjecture would solve a huge collection of famous and unsolved Diophantine equations in one stroke. That is because it would put explicit bounds on the size of the solutions. For example, *abc* might show that all the solutions to an equation must be smaller than 100. To find those solutions, all one would have to do would be to plug in every number from 0 to 99 and calculate which ones work. Without *abc*, by contrast, there would be infinitely many numbers to plug in.

Elkies's work meant that the *abc* conjecture could supersede the most important breakthrough in the history of Diophantine equations: confirmation of a conjecture formulated in 1922 by the US mathematician Louis Mordell, which said that the vast majority of Diophantine equations either have no solutions or have a finite number of them. That conjecture was proved in 1983 by German mathematician Gerd Faltings, who was then 28 and within three years would win a Fields Medal, the most coveted mathematics award, for the work. But if *abc* is true, you don't just know how many solutions there are, Faltings says, “you can list them all”.

Soon after Faltings solved the Mordell conjecture, he started teaching at Princeton University in New Jersey — and before long, his path crossed with that of Mochizuki.

Born in 1969 in Tokyo, Mochizuki spent his formative years

in the United States, where his family moved when he was a child. He attended an exclusive high school in New Hampshire, and his precocious talent earned him an undergraduate spot in Princeton's mathematics department when he was barely 16. He quickly became legend for his original thinking, and moved directly into a PhD.

People who know Mochizuki describe him as a creature of habit with an almost supernatural ability to concentrate. “Ever since he was a student, he just gets up and works,” says Minhyong Kim, a mathematician at the University of Oxford, UK, who has known Mochizuki since his Princeton days. After attending a seminar or colloquium, researchers and students would often go out together for a beer — but not Mochizuki, Kim recalls. “He's not introverted by nature, but he's so much focused on his mathematics.”

Faltings was Mochizuki's adviser for his senior thesis and for his doctoral one, and he could see that Mochizuki stood out. “It was clear that he was one of the brighter ones,” he says. But being a Faltings student couldn't have been easy. “Faltings was at the top of the intimidation ladder,” recalls Kim. He would pounce on mistakes, and when talking to him, even eminent mathematicians could often be heard nervously clearing their throats.

Faltings's research had an outsized influence on many young number theorists at universities along the US eastern seaboard. His area of expertise was algebraic geometry, which since the 1950s had been transformed into a highly abstract and theoretical field by Alexander Grothendieck — often described as the greatest mathematician of the twentieth century. “Compared to Grothendieck,” says Kim, “Faltings didn't have as much patience for philosophizing.” His style of maths required “a lot of abstract background knowledge — but also tended to have as a goal very concrete problems. Mochizuki's work on *abc* does exactly this”.

Single-track mind

After his PhD, Mochizuki spent two years at Harvard and then in 1994 moved back to his native Japan, aged 25, to a position at RIMS. Although he had lived for years in the United States, “he was in some ways uncomfortable with American culture”, Kim says. And, he adds, growing up in a different country may have compounded the feeling of isolation that comes from being a mathematically gifted child. “I think he did suffer a little bit.”

Mochizuki flourished at RIMS, which does not require its faculty members to teach undergraduate classes. “He was able to work on his own for 20 years without too much external disturbance,” Fesenko says. In 1996, he boosted his international reputation when he solved a conjecture that had been stated by Grothendieck; and in 1998, he gave an invited talk at the International Congress of Mathematicians in Berlin — the equivalent, in this community, of an induction to a hall of fame.

But even as Mochizuki earned respect, he was moving away from the mainstream. His work was reaching higher levels of abstraction and he was writing papers that were increasingly impenetrable to his peers. In the early 2000s he stopped venturing to international meetings, and colleagues say that he rarely leaves the Kyoto prefecture any more. “It requires a special kind of devotion to be able to focus over a period of many years without having collaborators,” says number theorist Brian Conrad of Stanford University in California.

Mochizuki did keep in touch with fellow number theorists, who knew that he was ultimately aiming for *abc*. He had next to no competition:

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To hear a podcast on Shinichi Mochizuki's proof, visit: go.nature.com/v6rfty7

most other mathematicians had steered clear of the problem, deeming it intractable. By early 2012, rumours were flying that Mochizuki was getting close to a proof. Then came the August news: he had posted his papers online.

The next month, Fesenko became the first person from outside Japan to talk to Mochizuki about the work he had quietly unveiled. Fesenko was already due to visit Tamagawa, so he went to see Mochizuki too. The two met on a Saturday in Mochizuki's office, a spacious room offering a view of nearby Mount Daimonji and with neatly arranged books and papers. It is "the tidiest office of any mathematician I've ever seen in my life", Fesenko says. As the two mathematicians sat in leather armchairs, Fesenko peppered Mochizuki with questions about his work and what might happen next.

Fesenko says that he warned Mochizuki to be mindful of the experience of another mathematician: the Russian topologist Grigori Perelman, who shot to fame in 2003 after solving the century-old Poincaré conjecture (see *Nature* 427, 388; 2004) and then retreated and became increasingly estranged from friends, colleagues and the outside world. Fesenko knew Perelman, and saw that the two mathematicians' personalities were very different. Whereas Perelman was known for his awkward social skills (and for letting his fingernails grow unchecked), Mochizuki is universally described as articulate and friendly — if intensely private about his life outside of work.

Normally after a major proof is announced, mathematicians read the work — which is typically a few pages long — and can understand the general strategy. Occasionally, proofs are longer and more complex, and years may then pass for leading specialists to fully vet it and reach a consensus that it is correct. Perelman's work on the Poincaré conjecture became accepted in this way. Even in the case of Grothendieck's highly abstract work, experts were able to relate most of his new ideas to mathematical objects they were familiar with. Only once the dust has settled does a journal typically publish the proof.

But almost everyone who tackled Mochizuki's proof found themselves floored. Some were bemused by the sweeping — almost messianic — language with which Mochizuki described some of his new theoretical instructions: he even called the field that he had created 'inter-universal geometry'. "Generally, mathematicians are very humble, not claiming that what they are doing is a revolution of the whole Universe," says Oesterlé, at the Pierre and Marie Curie University in Paris, who made little headway in checking the proof.

The reason is that Mochizuki's work is so far removed from anything that had gone before. He is attempting to reform mathematics from the ground up, starting from its foundations in the theory of sets (familiar to many as Venn diagrams). And most mathematicians have been reluctant to invest the time necessary to understand the work because they see no clear reward: it is not obvious how the theoretical machinery that Mochizuki has invented could be used to do calculations. "I tried to read some of them and then, at some stage, I gave up. I don't understand what he's doing," says Faltings.

Fesenko has studied Mochizuki's work in detail over the past year, visited him at RIMS again in the autumn of 2014 and says that he has now verified the proof. (The other three mathematicians who say they have corroborated it have also spent considerable time working alongside Mochizuki in Japan.) The overarching theme of inter-universal geometry, as Fesenko describes it, is that one must look at whole numbers in a different light — leaving addition aside and seeing the multiplication structure as something malleable and deformable. Standard multiplication would then be just one particular case of a family of structures, just as a circle is a special case of an ellipse.

Fesenko says that Mochizuki compares himself to the mathematical giant Grothendieck — and it is no immodest claim. "We had mathematics before Mochizuki's work — and now we have mathematics after Mochizuki's work," Fesenko says.

But so far, the few who have understood the work have struggled to explain it to anyone else. "Everybody who I'm aware of who's come close to this stuff is quite reasonable, but afterwards they become incapable of communicating it," says one mathematician who did not want his name to be mentioned. The situation, he says, reminds him of the *Monty Python* skit about a writer who jots down the world's funniest joke. Anyone who reads it dies from laughing and can never relate it to anyone else.

And that, says Faltings, is a problem. "It's not enough if you have a good idea: you also have to be able to explain it to others." Faltings says that if Mochizuki wants his work to be accepted, then he should reach out more. "People have the right to be eccentric as much as they want to," he says. "If he doesn't want to travel, he has no obligation. If he wants recognition, he has to compromise."

Edge of reason

For Mochizuki, things could begin to turn around later this year, when the Clay Mathematics Institute will host the long-awaited workshop in Oxford. Leading figures in the field are expected to attend, including Faltings. Kim, who along with Fesenko is one of the organizers, says that a few days of lectures will not be enough to expose the entire theory. But, he says, "hopefully at the end of the workshop enough people will be convinced to put more of their effort into reading the proof".

Most mathematicians expect that it will take many more years to find some resolution. (Mochizuki has said that he has submitted his papers to a journal, where they are presumably still under review.) Eventually, researchers hope, someone will be willing not only to understand the work, but also to make it understandable to others — the problem is, few want to be that person.

Looking ahead, researchers think that it is unlikely that future open problems will be as complex and intractable. Ellenberg points out that theorems are generally simple to state in new mathematical fields, and the proofs are quite short.

The question now is whether Mochizuki's proof will edge towards acceptance, as Perelman's did, or find a different fate. Some researchers see a cautionary tale in that of Louis de Branges, a well-established mathematician at Purdue University in West Lafayette, Indiana. In 2004, de Branges released a purported solution to the Riemann hypothesis, which many consider the most important open problem in maths. But mathematicians have remained sceptical of that claim; many say that they are turned off by his unconventional theories and his idiosyncratic style of writing, and the proof has slipped out of sight.

For Mochizuki's work, "it's not all or nothing", Ellenberg says. Even if the proof of the *abc* conjecture does not work out, his methods and ideas could still slowly percolate through the mathematical community, and researchers might find them useful for other purposes. "I do think, based on my knowledge of Mochizuki, that the likelihood that there's interesting or important math in those documents is pretty high," Ellenberg says.

But there is still a risk that it could go the other way, he adds. "I think it would be pretty bad if we just forgot about it. It would be sad." ■

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